INVARIANTS OF KNOT DIAGRAMS AND RELATIONS AMONG REIDEMEISTER MOVES

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ABSTRACT. In this paper a classification of Reidemeister moves, which is the most refined, is introduced. In particular, this classification distinguishes some Ω_3-moves that only differ in how the three strands that are involved in the move are ordered on the knot.

To transform knot diagrams of isotopic knots into each other one must in general use Ω_3-moves of at least two different classes. To show this, knot diagram invariants that jump only under Ω_3-moves are introduced.

Knot diagrams of isotopic knots can be connected by a sequence of Reidemeister moves of only six, out of the total of 24, classes. This result can be applied in knot theory to simplify proofs of invariance of diagrammatical knot invariants. In particular, a criterion for a function on Gauss diagrams to define a knot invariant is presented.

1. KNOT DIAGRAMS AND REIDEMEISTER MOVES.

1.1. **Knot diagrams.** A *knot diagram* is a picture of an oriented smooth knot like the *figure eight knot diagram* in Figure 1. Formally, this is the image of an immersion of S^1 in $\mathbb{R}^2$, with transversal double points and no points of higher multiplicity, which has been decorated so that we can distinguish:

a: An orientation of the strand, and
b: an overpassing and an underpassing strand at each double point.

Of course, the manner of decoration, such as where an arrow that indicates the orientation is placed, does not matter: Two knot diagrams are the same if they are made from the same image, have the same orientation, and have the same over-undercrossing information at each crossing point.

Two knot diagrams are *equivalent* if there is an orientation-preserving diffeomorphism of the plane that takes one diagram to the other diagram.

1.2. **Reidemeister moves.** A *Reidemeister move* is a transformation of a knot diagram which looks like one of the transformations in Table 1 inside a disk in $\mathbb{R}^2$ (up to diffeomorphisms) and leaves the knot diagram unchanged outside the disk. More formally, we say that the knot diagrams k and l are related by a Reidemeister
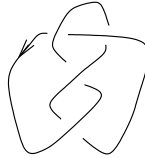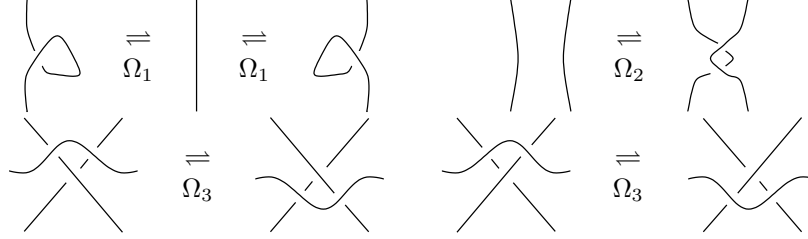
FIGURE 1. The figure eight knot diagram.

1

TABLE 1. Reidemeister moves.

move if they are the same outside $C \subset \mathbb{R}^2$ and there are diffeomorphisms g, h from C to a disk D such that

a: $h^{-1} \circ g = Id$ on the boundary of C, and

b: $g(k \cap C)$ and $h(l \cap C)$ is one of the unordered pairs of diagram-parts in Table 1.

We shall call C the *changing disk* of the Reidemeister move.

Reidemeister moves were introduced in the setting of PL-knots by K. Reidemeister [3] in 1927. Reidemeister proved the PL version of the following well-known fact: *Two knot diagrams describe isotopic knots if and only if they are connected by a sequence of Reidemeister moves and orientation-preserving diffeomorphisms of the plane.*

1.3. **Classifying Reidemeister moves.** Reidemeister moves are usually sorted into three classes Ω_1, Ω_2 and Ω_3, as in Table 1. That is, a move is of class Ω_i if the changing disk contains i strands. (Trivia: In his first paper on the subject [3], Reidemeister put the moves in another order: He called them *Operation* 1 (Ω_2), 2 (Ω_1) and 3 (Ω_3). The terminology $\{\Omega.1, \Omega.2, \Omega.3\}$ appears in his book "Knotentheorie" [4] from 1932).

A more refined classification of Reidemeister moves may be needed when one wants to prove that a function defined on knot diagrams is a knot invariant. There are two common ways in which the $\{\Omega_1, \Omega_2, \Omega_3\}$-classification can be refined: One is to take into account *the orientation of* $\mathbb{R}^2$ (that is, to distinguish the mirror-imaged versions of Ω_1- and Ω_3-moves in Table 1). The other is to also take into account *the orientations of the strands in the changing disk*. In this paper a classification is introduced (see Definition 1 below) which incorporates these refinements and also the *ascendingness* and *descendingness* defined in Section 1.4 below. The classification in Definition 1 originated in a study of *Gauss diagrams*. In Section 2 the classification is reformulated in terms of Gauss diagrams.

1.4. **Ascending and descending Ω_3-moves.** The original part of the classification in Definition 1 is contained in the following characterization of an Ω_3-move. We can distinguish the three strands in the changing disk of an Ω_3-move: One is the *top* strand, the strand with two over-crossings. One is the *middle* strand, and one the *bottom* strand. When we move along the knot diagram in the direction of the orientation, we pass these strands in some cyclic order. Call the Ω_3-move *descending* if we pass the strands in the order *top-middle-bottom* and call it *ascending* otherwise.

Theorem 1. *The figure eight knot diagram cannot be transformed into its inverse (the same diagram with the orientation of the strand reversed) without the use of both ascending and descending Ω_3-moves. In particular, it cannot be transformed into its inverse without at least two Ω_3-moves.*

The proof is given in Section 5.

1.5. **A refined classification of Reidemeister moves.** By definition, a knot diagram is made from some immersion of S^1 in $\mathbb{R}^2$. We shall assume that the orientation of the knot diagram is induced from the orientation of S^1, so the knot diagram determines the immersion up to an orientation-preserving diffeomorphism of S^1.

Let the knot diagrams d, k be decorated images of the immersions $i_d, i_k : S^1 \to \mathbb{R}^2$. Given these immersions, an equivalence of the knot diagrams d and k (i.e. an orientation-preserving diffeomorphism of $\mathbb{R}^2$ taking d to k) induces an orientation-preserving diffeomorphism of S^1.

Definition 1 (Equivalence of Reidemeister moves). *Let X be the Reidemeister move that transforms the knot diagram d_1 to d_2 inside the changing disk C, and let Y be the Reidemeister move that transforms k_1 to k_2 inside the changing disk D.*

X and Y are directed-equivalent if there are orientation-preserving diffeomorphisms $g, h : C \to D$ such that

a) *$g(d_1 \cap C) = k_1 \cap D$ and $h(d_2 \cap C) = k_2 \cap D$, with the same over-undercrossing information at each crossing.*

b) *Fix some immersions of the oriented circle such that d_1, d_2, k_1, k_2 are their decorated images. The maps g, h of pieces of knot diagrams induce maps between segments of the immersing circles. These maps should extend to orientation-preserving diffeomorphisms $S^1 \to S^1$.*

X and Y are equivalent if X is directed-equivalent either to Y or to the reverse move Y^{-1} that changes k_2 into k_1 inside D.

Every equivalence class splits into exactly two directed equivalence classes; see Section 1.8.

Each crossing in a knot diagram is given a *sign* by the rule $\text{sign}(\!\overset{\displaystyle |}{\underset{\displaystyle \downarrow}{\longleftarrow}}\!) = +1$,

$\text{sign}(\!\overset{\displaystyle \uparrow}{\underset{\displaystyle |}{\longleftarrow}}\!) = -1$.

Definition 2 (Taxonomy of Reidemeister moves).

- *An Ω_1-move is said to be of class Ω_{1ij}, $i, j = \pm 1$. The index $i = +1$ if the kink that the move introduces (or removes) is oriented counterclockwise, and $i = -1$ otherwise. The index j is the sign of the crossing that the move introduces or removes.*

- *An Ω_2-move is said to be of class Ω_{2ij}, $i, j = \pm 1$. The index $i = +1$ if the meeting strands are oriented in the same direction (the move is called direct), and $i = -1$ otherwise (the move is called inverse). The index j is the sign of the affected crossing where the overcrossing strand is directed out from the area enclosed by the meeting strands:*

- *An Ω_3-move is said to be of class Ω_{3ijkm}, $i, j, k, m = \pm 1$. The index $m = +1$ if the move is descending and $m = -1$ otherwise. The indices i, j, k are the signs of the crossings in the changing disk:*
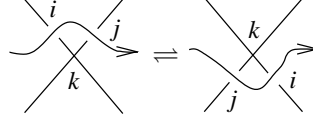
It is easy to verify that Definition 2 gives the splitting of the classes Ω_1, Ω_2 and Ω_3 into equivalence classes as defined in Definition 1.

1.6. **Main results on classes of Reidemeister moves.**

Theorem 2. *Any Reidemeister move can be realized by a sequence of moves of the classes Ω_{1++}, Ω_{1+-}, Ω_{2-+}, Ω_{2--}, Ω_{3-+++}, Ω_{3--+-}, followed by a diffeomorphism with support in the changing disk of the original move.*

The proof is given in Section 3. There are other sets of six classes of Reidemeister moves that also has this property, such as the set $\{\Omega_{1-+}$, Ω_{1--}, Ω_{2-+}, Ω_{2+-}, Ω_{3+--+}, $\Omega_{3++--}\}$.

Theorem 3. *For every knot type there are two knot diagrams, representing knots of this type, such that to transform them into each other we need at least*

1. *one ascending and one descending Ω_3-move, and*
2. *an Ω_{1ij}-move and an Ω_{1kl}-move, $(i, j) \neq (k, l)$, $(i, j) \neq (-k, -l)$.*

Theorem 3 extends Theorem 1 to diagrams of any knot type. The proof is given in Section 5, and it makes use of some *knot diagram invariants* introduced in Section 4. If we remove requirement 1, the remaining statement can be proved using well-known knot diagram invariants (writhe and winding number, also cf. Proposition 2).

1.7. **Organization of the rest of the paper.** Sections 1.8 and 1.9 deal with the *direction* of a move. In Section 2 the definitions and results are reformulated in terms of *Gauss diagrams*. Section 3 contains the proof of Theorem 2.

In Section 4 several *knot diagram invariants* are introduced, and some of their properties are derived. Section 4.4 describes how the simplest of the invariants jumps under a Reidemeister move. Section 5 contains the proofs of Theorems 1 and 3. The proofs make use of some of the invariants from Section 4.

1.8. **Directed classes of Reidemeister moves.** Every equivalence class splits into exactly two directed equivalence classes (see Definition 1). We say that an Ω_1- or Ω_2-move is in *positive direction* if the number of crossings in the diagram increases. To define the positive direction of an Ω_3-move we use a construction by V. Arnold [1]:

The three strands in the changing disk (before or after the move) form a *vanishing triangle*. The edges are ordered by the cyclic order in which we pass them if we move around the diagram in the direction of the orientation. This cyclic order gives an orientation of the vanishing triangle. Let n be the number of edges where this orientation coincides with the orientation of the edge. Let $q = (-1)^n$. Then $q = -1$ at one side of the move and $q = +1$ on the other. We say that the move is in positive direction if $q = -1$ before the move and $q = +1$ after the move.

Example 1 (An Ω_3-move in positive direction)**.**
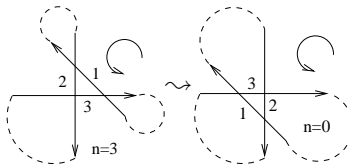
1.9. **The direction of ascending and descending Ω_3-moves.** Arnold's definition of the positive direction of a triple point move appeared in the theory of plane curves. For knot diagrams there is an ordering of the edges in the vanishing triangle which only depends on the knot diagram inside the changing disk: *top-middle-bottom.* So it is possible to define a "positive direction" of an Ω_3-move in two different ways: Using Arnold's ordering of the edges, or the top-middle-bottom ordering. The two definitions give the same "positive direction" for descending Ω_3-moves and opposite "positive direction" for ascending Ω_3-moves. (In this paper we always use Arnold's original definition.)

2. REIDEMEISTER MOVES OF GAUSS DIAGRAMS.

Let the knot diagram d be the decorated image of an immersion of an oriented circle, with the orientation of the diagram given by the orientation of the circle. The *Gauss diagram G_d* of d is the immersing oriented circle with the preimages of each crossing point connected by a signed arrow. The arrow points from overpass to underpass and is given the sign of the crossing. Gauss diagrams are considered up to orientation-preserving diffeomorphism of the circle, so equivalent knot diagrams have the same Gauss diagram.

Example 2 (The Gauss diagram of a knot diagram.)**.**

The Gauss diagram determines a knot diagram on the sphere up to an orientation-preserving diffeomorphism of the sphere. That is, the Gauss diagram determines a knot diagram in the plane up to a choice of a point of infinity outside the diagram on the sphere and up to equivalence of knot diagrams.

A Reidemeister move induces a transformation of Gauss diagrams. An Ω_i-move affects i arrows, with end points on i segments of the circle of the Gauss diagram, without changing the rest of the diagram.

The original motivation for the choice of the definitions in this paper was the following obvious Proposition:

Proposition 1. *The Reidemeister moves $X : d_1 \mapsto d_2$ and $Y : k_1 \mapsto k_2$ are directed-equivalent if and only if the abstract Gauss diagrams obtained by erasing all arrows in G_{d_1} and G_{k_1} that are not affected by the move are the same (up to an orientation-preserving diffeomorphism of the circle), and likewise for G_{d_2} and G_{k_2}.*
□

That is, X and Y are directed-equivalent if and only if the induced transformations of the Gauss diagrams look the same when disregarding the unchanged part of the diagram.

TABLE 2. A sufficient set of Reidemeister moves for Gauss diagrams.
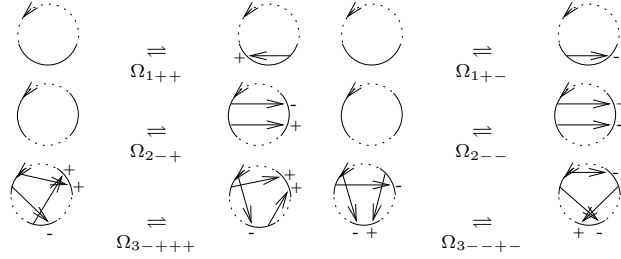
Corollary 1. *A function on Gauss diagrams defines a knot invariant if and only if it is invariant under the transformations in Table 2, where the dotted segments indicate a part of the Gauss diagram that is unchanged.*

In Table 2, the positive direction of a move is from left to right.

3. PROOF OF THEOREM 2.

The proof of Theorem 2 consists of three lemmas. The method is to draw movies that show explicitly how a Reidemeister move of a class we want to exclude is replaced by a sequence of other Reidemeister moves. All moves in this sequence take place inside the changing disk of the move we replace.

Lemma 1. *Any Ω_1-move can be replaced by Ω_{1++}-, Ω_{1+-}- and Ω_{2--}-moves.*

Proof: We can replace an Ω_{1-+}-move with the sequence
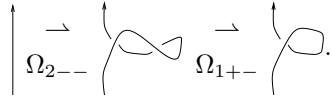
And we can replace an Ω_{1--}-move with the sequence

□
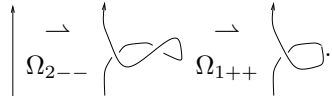
Lemma 2. *Any direct Ω_2-move can be replaced by Ω_{2--}-, Ω_{1-+}-, Ω_{1--}-, Ω_{3-+++}- and Ω_{3--+-}-moves.*
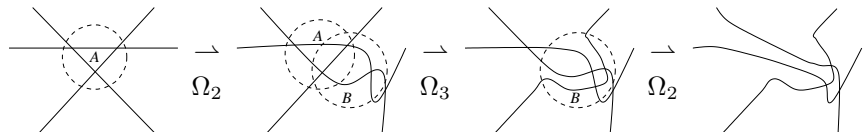
Proof: We can replace an Ω_{2++}-move with the sequence
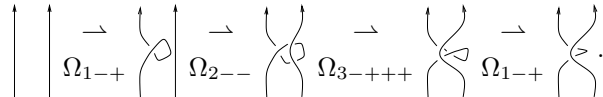
In a similar way we can replace an Ω_{2+-}-move with a sequence of moves of class Ω_{1--}, Ω_{2--}, Ω_{3--+-} and Ω_{1--}.

□

Lemma 3. *Any ascending (descending) Ω_3-move can be replaced by Ω_2-moves and an Ω_3-move of any ascending (descending) class.*

Proof: We can replace an Ω_3-move with a sequence which looks like

if we temporarily forget about orientation, over/under-crossing information and the ordering of the edges. Here we replace the Ω_3-move in the changing disk A with a move in the changing disk B. The Ω_3-moves in A and in B are either both ascending or both descending. The first three signs i, j, k in the Ω_{3ijkm} index are given by the signs of the three crossings in the changing disk as indicated in Definition 2.

We have three choices of wedges where to make the first Ω_2-move. If we make the move between the middle and bottom strands, i and j change place in the index and k is reversed: $(i, j, k) \mapsto (j, i, -k)$. If we make the move between the top strand and the strand that crosses the top strand at the i-crossing then $(i, j, k) \mapsto (j, -i, k)$. If we make the move between the top strand and the strand that crosses the top strand at the j-crossing we get $(i, j, k) \mapsto (-j, i, k)$. Repeating this process with the new changing disk, we can get any index. This completes the proof of Theorem 2. □

## 4. Knot diagram invariants.

4.1. **Writhe and winding number.** Note that by a *knot diagram invariant* we mean a function on knot diagrams that is unchanged under equivalence of knot diagrams. There are some well-known knot diagram invariants which are not knot invariants, foremost the *writhe* and the *winding number*. The writhe is the sum of the signs of all crossings in the diagram. The winding number is the degree of the map $S^1 \to S^1$ taking a point on the immersing circle to the direction of the tangent vector in the plane.

The main previously known result about classes of Reidemeister moves is Proposition 2 below. Note that the writhe and the winding number are unchanged under Ω_1- and Ω_2-moves, which proves the easy "only if"-part of the Proposition.

Proposition 2 (B. Trace [5]). *Two knot diagrams of isotopic knots can be transformed into each other without the use of Ω_1-moves (they are* regularly isotopic*) if and only if they have the same writhe and winding number.*

4.2. **Connected sum of knot diagrams.** The knot diagram k is a *connected sum* of k_1 and k_2 if k can be split by a simple closed curve intersecting the diagram in two points, into diagrams equivalent to k_1 and k_2, as below:
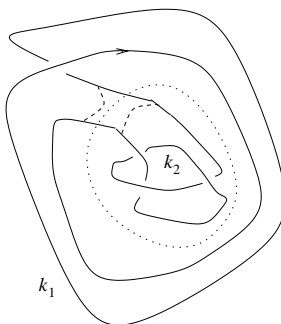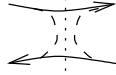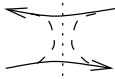
If k is a connected sum of k_1 and k_2, the corresponding Gauss diagram G_k is a connected sum of G_{k_1} and G_{k_2} in the sense that G_k can be split into G_{k_1} and G_{k_2}: In the example above, the terms have Gauss diagrams $G_{k_1} = $ and $G_{k_2} = $,

while the sum has Gauss diagram . A knot diagram invariant V is called *additive* if $V(k) = V(k_1) + V(k_2)$ whenever k is a connected sum of k_1 and k_2.

Lemma 4. *The writhe is additive. The winding number is not, but if k is a connected sum of k_1 and k_2, then*

$$winding\ number(k) = winding\ number(k_1) + winding\ number(k_2) + i$$

where i depends on how the sum is done: $i = +1$ if we cut like *and*

$i = -1$ *if we cut like* .

Proof: The first part is evident. For the second part: If we cut the knot diagram k into k_1 and k_2 we introduce two new half-turns of the direction vector, in the same direction. Hence the sum of the winding numbers of k_1 and k_2 differ by $+1$ from the winding number of their connected sum k if the half-turns are in positive direction, and by -1 if they are in negative direction. $\square$

4.3. **The knot diagram invariants A_n, D_n and W_n.** For a knot diagram k let $A_4(k)$ be the sum of the *signs* of all subdiagrams of the form $a_4 = $ in the Gauss diagram G_k of k. A subdiagram is an abstract Gauss diagram that can be created from G_k by removing arrows. The sign of the subdiagram is the product of the signs of the arrows in the subdiagram. In the same way define D_4 as the sum of the signs of the subdiagrams of the form $d_4 = $. In Section 5 we use A_4 and D_4 to prove Theorems 1 and 3.

More generally, we define $A_n(k)$ and $D_n(k)$, $n \geq 4$, as the sum of the signs of the subdiagrams of G_k with n arrows arranged cyclically as

respectively

.

Define $W_n(k)$, $n = 3, 5, 7, \ldots$, as the sum of the signs of the subdiagrams of G_k with n arrows which are diameters directed so that no two arrow heads are adjacent on the circle. That is, W_5 is the sum of the signs of the subdiagrams that look like

$w_5 = $.

Obviously, A_n, D_n and W_n are knot diagram invariants. The abstract subdiagrams that define these invariants, which we call a_n, d_n and w_n, are examples of *arrow diagrams.* Functions defined in this way were introduced by M. Polyak and O. Viro [2].

Proposition 3 (Properties of A_n, D_n and W_n)**.** *A_n, D_n and W_n are additive under the connected sum of knot diagrams. $A_n(k)$ and $D_n(k)$ are multiplied by $(-1)^n$ by mirroring of k (change of orientation of $\mathbb{R}^2$) and change into each other under inversion of k (change of orientation of the strand). $W_n(k)$ is multiplied by -1 by mirroring of k and is unchanged by inversion of k.*

Proof: Mirroring of the knot diagram affects the Gauss diagram by reversing the signs of all arrows. The sign of an a_n-, d_n- or w_n-subdiagram is a product of the signs of n arrows, so every sign is multiplied by $(-1)^n$. For a w_n-subdiagram, n is always odd.

Inversion of the knot diagram affects the Gauss diagram by reversing the orientation of the circle, so the statement is evident.

Additivity is because the Gauss diagram of a connected sum is a connected sum of Gauss diagrams (see Section 4.2), and each a_n-, d_n-, or w_n-subdiagram evidently must belong to one of the summands. $\qquad\square$

Proposition 4 (Reidemeister move invariance of A_n, D_n and W_n)**.** *A_n, D_n and W_n do not change under Ω_1- and Ω_2-moves. A_n does not change under descending Ω_3-moves and D_n does not change under ascending Ω_3-moves.*

We assert that A_n (D_n) does change under some ascending (descending) Ω_3-moves and W_n does change under some Ω_3-moves of both ascending and descending type. (That is, they are not knot invariants.)

Proof of Ω_1-invariance: An Ω_1-move introduces (or removes) one arrow in the Gauss diagram as in Table 2. This arrow is *isolated*; its head and tail are adjacent on the circle. a_n-, d_n- and w_n-subdiagrams contain no such arrows, so the set of such subdiagrams does not change. $\qquad\square$

Proof of Ω_2-invariance: An Ω_2-move introduces (or removes) two arrows in the Gauss diagram as in Table 2. The tails of these arrows are adjacent on the circle, and the heads are also adjacent on the circle. This is not the case for any two arrows in an a_n-, d_n- or w_n-subdiagram. There may be a_n-, d_n- or w_n-subdiagrams that contain one of the two arrows affected by the move. However, such subdiagrams come in pairs with opposite sign, since the two arrows have opposite sign and point from adjacent points on the circle to adjacent points on the circle. Hence A_n, D_n and W_n are invariant under Ω_1- and Ω_2-moves. $\qquad\square$

Proof of invariance of A_n (D_n) under descending (ascending) Ω_3-moves: By Lemma 3, a knot diagram invariant that is unchanged under Ω_1- and Ω_2-moves is invariant under all ascending (respectively descending) Ω_3-moves if it is invariant under moves of class Ω_{3--+-} (Ω_{3-+++}). The corresponding moves of Gauss diagrams are depicted in Table 2.

An Ω_3-move affects three arrows in the Gauss diagram. The move obviously does not change the number of a_n- and d_n-subdiagrams that only contain one of these three arrows. The subdiagrams that contain two of the affected arrows might change. These two arrows must then have adjacent end points at some point on the circle.

Any two arrows in an a_n-subdiagram have adjacent end points at no more than one place on the circle. Then these end points are one head and one tail. If the two arrows are crossed, the tail comes before the head on the circle, and if the arrows are not crossed, the head comes before the tail.

We see in Table 2 that for an Ω_{3-+++}-move, two affected arrows that have a head and a tail next to each other are crossed if and only if the head comes before the tail. Hence no pair of arrows that are affected by an Ω_{3-+++}-move can belong to an a_n-subdiagram. Consequently A_n is invariant under descending Ω_3-moves. In the same way we see that no pair of arrows that are affected by an Ω_{3--+-}-move can belong to a d_n-subdiagram.   □

4.4. The Weirdness W_3. A geometric formula for the jump. The function

W_3, which is given by subdiagrams on the form $w_3 = \left(\!\!\!\begin{array}{c}\otimes\end{array}\!\!\!\right)$, is called *Weirdness*.
We shall give a formula for the jump of W_3 under a positively directed Ω_3-move.

Let k be a knot diagram and C a selected changing disk for a Ω_3-move on k. The piece of knot diagram exterior to C consists of three strands, which we label a, b and c. a is the strand that connects the top strand inside C with the middle strand. b connects the middle and the bottom strand, and c connects the top and the bottom strand. By the *local linking number llk* of the pair (k, C) we mean the following (cf. Example 3 below):

Erase the piece of knot diagram that lies inside C. Erase the c strand. Make the remaining a and b strands into a two-component link diagram by connecting the two ends of each strand with a strand inside C in such a way that the strand that closes the b strand *does not pass over* the strand that closes the a strand.

The local linking number is the linking number of this link. (The linking number is an invariant of two-component links. If we pick one of the components, the linking number is the sum of the signs of the crossings where this component crosses over the other.)

llk is clearly invariant under equivalence of knot diagrams (as long as we keep track of the selected changing disk.)

Proposition 5. W_3 *jumps by* $llk(k, C)$ *under a positively directed Ω_3-move in the changing disk C on the knot diagram k.*

Example 3 (Computing the local linking number.). Consider the changing disk C in the knot diagram
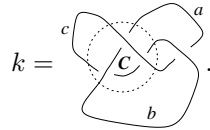

We make the knot diagram into a link diagram as below:
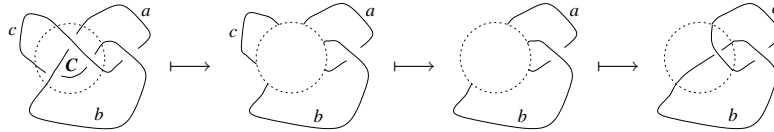
llk is the linking number of this link.

Proof of Proposition 5: By Lemma 3, we can replace an Ω_3-move with a sequence consisting of Ω_2-moves and a move of any Ω_3-class with the same ascending-ness/descendingness. The main trick is to make a move in the changing disk B

instead of the changing disk A in the picture below:

$$k_1 = \qquad \xrightarrow{\ \Omega_2\ } \qquad = k_2$$
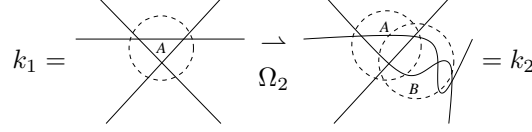
Here $llk(k_1, A) = llk(k_2, A)$ since Ω_2-moves outside A cannot change the linking number. $llk(k_2, A) = llk(k_2, B)$ since the most that can happen is that an overcrossing of the a (or b) strand is replaced with an overcrossing of the b (a) strand with opposite sign. This does not change the linking number. Hence it is sufficient to prove the statement for Ω_{3-+++}-moves and Ω_{3--+-}-moves. We consider w_3-subdiagrams of a Gauss diagram, such that two of the arrows in the subdiagrams belong to the changing part. (Subdiagrams such that only one arrow belong to the changing part are unchanged by the move.) The changing part is depicted in Table 2. We see that there can be no subdiagrams of this kind after the positively directed move, when the changing part looks as on the right side in the Table. Before the positively directed Ω_{3-+++}-move we can have w_3-subdiagrams that look like , which contains the two dotted arrows in the changing part and one arrow pointing between unchanged segments as shown. Before the positively directed Ω_{3--+-}-move we can have w_3-subdiagrams that look like .
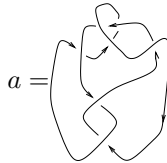
In both cases, the sign of the subdiagram is minus the sign of the third arrow in the subdiagram. The third arrow points from the segment that represents the b strand to the segment that represents the a strand. So when we make the move in positive direction, we add to W_3 the sum of the signs of the crossings where the b strand crosses over the a strand. This is the local linking number, since we close the link without introducing any new overcrossings of b over a. $\qquad\square$

5. Proof of Theorems 1 and 3.

5.1. **Proof of Theorem 1.** We shall prove that the figure eight diagram d_8 in Figure 1 cannot be transformed into its inverse without the use of both ascending and descending Ω_3-moves. d_8 has Gauss diagram $G_{d_8} =$. Obviously $A(d_8) = 1$ while $D(d_8) = 0$. The inverse knot diagram has $A = 0$ and $D = 1$ by Proposition 3. The result follows since A jumps only under ascending Ω_3-moves and D jumps only under descending Ω_3-moves. $\qquad\square$

5.2. **Proof of Theorem 3.** For each knot diagram, we shall construct a knot diagram of an isotopic knot, such that the two knot diagrams can only be transformed into each other with the use of both descending and ascending Ω_3-moves, and move of some classes Ω_{1ij} and Ω_{1kl}, $(i, j) \neq \pm(k, l)$. Consider the knot diagram a of the unknot:

$$a =$$

a has Gauss diagrams $G_a = $. We see that $D(a) = -1$, $A(a) = 0$, writhe$(a) = -1$ and winding number$(a) = 0$. The inverse knot diagram $\tilde{a}$ has $D(\tilde{a}) = 0$, $A(\tilde{a}) = -1$, writhe$(a) = -1$ and winding number$(a) = 0$. (The writhe is unchanged under inversion, and the winding number is multiplied with -1.) For every knot diagram d there is some knot diagram k that is a connected sum of d, a and $\tilde{a}$. This connected sum should be made at corresponding points on a and $\tilde{a}$.

k and d describe isotopic knots, since we have just added two unknots. $A(k) = A(d) - 1$ and $D(k) = D(d) - 1$ by Proposition 3, hence we need both ascending and ascending Ω_3-moves to transform k to d. writhe$(k) = $ writhe$(d) - 2$ and winding number$(k) = $ winding number(d) by Lemma 4 (because the contribution in winding number from a and $\tilde{a}$ cancel). The winding number and writhe jump by $\pm i$ and $\pm j$ under an Ω_{1ij}-move and are invariant under Ω_2- and Ω_3-moves, so a collection of moves of classes Ω_{1ij} and $\Omega_{1,-i,-j}$ can never raise the writhe by 2 and leave the winding number unchanged. $\qquad\square$

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